

A note on LBI Tests for Normality in Exponential Power Distributions ¹

Sui Test di Normalità Invarianti Localmente Ottimi nella classe delle distribuzioni "exponential power"

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Riassunto: Questo lavoro presenta una derivazione alternativa del test invariante localmente ottimo (LBI) per la normalità nella classe delle normali di ordine r , piu' note come exponential power distributions. Rispetto ad altri metodi ricorrenti in letteratura, la strada qui seguita offre una prospettiva computazionale piu' semplice ed una interpretazione unitaria dei risultati ottenuti; si apre inoltre ad uno sviluppo bayesiano dell'analisi del problema della curtosi.

Keywords: Locally best invariant test, test for normality, exponential power distributions, Bayesian diagnostic for kurtosis.

1. Introduction and background.

Let us consider a family of probability density functions $g_\theta(y|a, b)$, where a and b are location and scale parameters, and θ is a shape parameter,

$$g_\theta(y|a, b) = bg_\theta(by - a), \quad -\infty < a < \infty, \quad b > 0, \quad \theta \in \Theta.$$

Let us assume, in addition, that there exists a particular value of θ , θ_0 say, corresponding to which it holds $g_{\theta_0}(by - a) = N(0, 1)$, where $N(0, 1)$ is the standard normal distribution. Then, given a random sample $y_1, \dots, y_n$, we can obtain a location-scale invariant test for $H_0 : \theta = \theta_0$ against the one-sided alternative $H_1 : \theta > \theta_0$ (or $H_1 : \theta < \theta_0$) by using the so-called marginal likelihood

$$L(\theta|y_1, \dots, y_n) = \int \prod_{i=1}^n g_\theta(by_i - a) b^{n-2} da db, \quad (1)$$

(see Hájek and Šidák, 1967, pp. 45-49). On the one hand, the best invariant test is based on the marginal likelihood ratio statistic (but often it does depend on the unknown value of θ under the alternative hypothesis); on the other hand, the locally best invariant (LBI) test rejects H_0 for large values of the efficient score,

$$\left. \frac{d \log(L(\theta|y_1, \dots, y_n))}{d\theta} \right|_{\theta=\theta_0} > C_\alpha \quad (\text{or } < C_\alpha), \quad (2)$$

¹The first idea of this paper originated from the discussion of the article: Chiodi M. (2000) Le curve normali di ordine p nell'ambito delle distribuzioni di errori accidentali: una rassegna dei risultati e problemi aperti per il caso univariato e per quello multivariato in *Atti della XL Riunione Scientifica della S.I.S.*, 289-300.

with C_α chosen in order to obtain the prescribed significance level α (see Spiegelhalter, 1983, Diana and Pace, 1985).

When g_θ is the exponential power distribution class (for a rich list of references see Di Ciccio and Monti, 2004) the LBI test for normality has been derived in an approximated form by Salvan (1986). The exact form is given in a multivariate context by Kuwana and Kariya (1991) together with its null and nonnull asymptotic distributions when the location parameter is assumed to be known; while the difficulties associated with an unknown location parameter are avoided by introducing a "naturally" modified version of the LBI test in the previous case. The problem left open in this paper is solved by Kuwana and George (1995) in the family of "curved" alternatives to the normal distribution (a wider class including the exponential power distributions as a special case): the general form of LBI test is seen to be an efficient score conditional on a and b averaged according to given weighting functions ((2.5), p. 443) and its asymptotic properties are presented. In all these papers, though the exponential power distribution family is written according to different choices of the parameters (a, b, θ) , the LBI test is derived by interchanging differentiation and integration signs in (2). Differently, in Spiegelhalter (1983, sec. 5) an asymptotically most powerful test is based on the "approximate" efficient score, which is an efficient score for known values of a and b with a and b taken to be equal to their estimates under H_0 . Here we express the exponential power distribution class as in Box and Tiao (1973) and derive the exact form of the LBI test. The obtained result is at the same time a more simple averaged efficient score and a naturally modified version of the LBI test in the case of known location parameter (section 2). Moreover, the LBI in our form can be interpreted as an upper bound for a Bayesian diagnostic (Carota, 2005) for departures from normality in the same class of alternatives (section 3).

2. The LBI test.

Let us embed the null Gaussian distribution in the class of exponential power distributions as follows

$$f(y|\mu, \sigma, \beta) = \omega(\beta)\sigma^{-1}\exp\left(c(\beta)\left|\frac{y-\mu}{\sigma}\right|^{\frac{2}{\beta+1}}\right) \quad -\infty < y < \infty$$

where

$$c(\beta) = \left[\frac{\Gamma(\frac{3}{2}(1+\beta))}{\Gamma(\frac{1}{2}(1+\beta))}\right]^{\frac{1}{1+\beta}}, \quad \omega(\beta) = \frac{[\Gamma(\frac{3}{2}(1+\beta))]^{\frac{1}{2}}}{(1+\beta)[\Gamma(\frac{1}{2}(1+\beta))]^{\frac{3}{2}}},$$

and $\sigma > 0$, $-\infty < \mu < \infty$, $-1 < \beta \leq 1$ (Box and Tiao, 1973).

The parameters μ and σ are respectively the mean and the standard deviation, while β controls the kurtosis of the distribution: for $\beta < 0$ f is platykurtik, for $\beta > 0$ it is leptokurtik and when $\beta = 0$ it reduces to $N(\mu, \sigma)$. Note moreover that $f(y|\mu, \sigma, \beta)$ is the specification of g corresponding to $(a, b, \theta) = (\mu/\sigma, 1/\sigma, \beta)$, while the choices of parameters in Salvan (1986) and in Kuwana and Kariya (1991) respectively are: $(a, b, \theta) = (\mu/\sigma_r, 1/\sigma_r, r)$ where $\sigma_r = [E(|y - \mu|^r)]^{1/r}$, $r = 2/(1 + \beta)$, and $(a, b, \theta) = (\mu/\phi, 1/\phi, \gamma)$ where $\phi = \sigma[2c(\beta)]^{-(1+\beta)/2}$, $\gamma = 1/(1 + \beta)$. Finally, with respect to this last case, in Kuwana

and George (1995) and in Spiegelhalter (1983) only the shape parameter is slightly modified: $\theta_{KG} = \gamma - 1$ and $\theta_{Sp} = \gamma/2$, respectively.

The likelihood function of (μ, σ, β) is

$$L(\mu, \sigma, \beta | y_1, \dots, y_n) = \omega(\beta)^n \sigma^{-n} \exp \left(c(\beta) \sum_{i=1}^n \left| \frac{y_i - \mu}{\sigma} \right|^{\frac{2}{\beta+1}} \right),$$

and the marginal likelihood of β turns out to be

$$L(\beta | y_1, \dots, y_n) = \int L(\mu, \sigma, \beta | y_1, \dots, y_n) \sigma^{-1} d\sigma d\mu \propto \frac{\Gamma(1 + \frac{n}{2}(1 + \beta)) J(\beta)}{[\Gamma(1 + \frac{1}{2}(1 + \beta))]^n}$$

with

$$J(\beta) = \int \left[\sum_{i=1}^n |y_i - \mu|^{\frac{2}{\beta+1}} \right]^{-\frac{n}{2}(1+\beta)} d\mu.$$

Then we can obtain the LBI test $\left. \frac{d \log(L(\beta | y_1, \dots, y_n))}{d\beta} \right|_{\beta=0} =$

$$\frac{n}{2} \left[\Psi \left(\frac{n}{2} \right) - \Psi \left(\frac{1}{2} \right) - \log(n) \right] - (n-1) + \frac{1}{2} \sum_{i=1}^n E_W [z_i^2 \log(z_i^2)], \quad (3)$$

where $z_i = \frac{|y_i - \mu|}{(\sum_{i=1}^n |y_i - \mu|^2 / n)^{\frac{1}{2}}}$ and $W = t(\bar{y}, \frac{S^2}{n}, n-1)$ is a Student t distribution with $n-1$ degree of freedom and location and scale parameters respectively given by the sample mean $\bar{y}$ and the sample variance $S^2 = \sum (y_i - \bar{y})^2 / (n-1)$ divided by n .

With the choice of parameters (μ, σ, β) , we interchange differentiation and integration signs only with respect to μ (when considering $J'(\beta)/J(\beta)|_{\beta=0}$) and the LBI test turns out to be an efficient scores for a known value of the mean averaged according to W , that is, a naturally modified version of the LBI test in this case. More precisely, when μ is known our formula (3) coincides up to the sign (ascribable to the different choice of θ) with (2.4) in Kuwana and Kariya (1991), while, when μ is unknown, (3) differs from their formula (4.5) (although they are the same test) and meet the intuitive idea expressed by their modified test (4.16). When μ is known, moreover, (3) coincides with the result in Spiegelhalter (1983).

3. LBI tests and Bayesian analysis of kurtosis

From a Bayesian point of view, the weighting function W can be interpreted as the posterior distribution of μ conditional on $\beta = 0$ and corresponding to the non informative reference prior $p(\mu, \sigma | \beta) \propto 1/\sigma$. If one accepts such an interpretation, the LBI test enters, in the role of upper bound, a Bayesian analysis of kurtosis based on the symmetric diagnostic (Carota, 2005)

$$\bar{\Delta}_L = \frac{1}{2} [E(\beta | y) - E(\beta)] \left. \frac{d \log(L(\beta | y_1, \dots, y_n))}{d\beta} \right|_{\beta=0} \quad (4)$$

where $E(\beta|y)$ and $E(\beta)$ respectively are the posterior and the prior expected values of the kurtosis parameter. In fact, $[E(\beta|y) - E(\beta)] \leq 2$ independently of which prior β is assigned.

In our kind of analysis, however, it is "natural" to consider priors expressing an high degree of belief in the null hypothesis, that is priors peaked around the null value $\beta_0 = 0$. As in Box and Tiao (1973) we adopt the prior

$$\pi(\beta) = \Gamma(2\phi)[\Gamma(\phi)]^{-2}2^{-2\phi-1}(1 - \beta^2)^{\phi-1}, \quad \beta \in (-1, 1], \quad \phi \geq 1$$

and consider the Darwin data (see Table (3.2.1), p.153, and fig. (3.2.5), p.166), but we focus on the Bayesian diagnostic (4), whereas they discuss robust inferences for the location parameter. If, in order to make a comparison, we assume that the value of μ is known and given by the value of the sample mean, 20.9333, the Δ_L -values corresponding to $\phi = 1, 6, 16$ respectively are of : 0.41, 0.28 and 0.037. According to the calibration in Carota (2005) none of them strongly induces to reject the normal model and for $\phi \geq 16$ they become very favourable to it. The same data set is analyzed by Spiegelhalter (1983) and its approximate efficient score statistic is seen to lie just inside the upper 95% point of the estimated null distribution.

Finally note that this analysis of kurtosis can be readily extended to the residuals of a fitted linear or non linear regression.

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